A LESS RESTRICTIVE BRIANÇON-SKODA THEOREM WITH COEFFICIENTS

IAN M. ABERBACH AND ALINE HOSRY

Abstract. The Briançon-Skoda theorem in its many versions has been studied by algebraists for several decades. In this paper, under some assumptions on an F-rational local ring $(R, \mathfrak{m})$, and an ideal I of R of analytic spread ℓ and height $g < \ell$, we improve on two theorems by Aberbach and Huneke. Let J be a reduction of I. We first give results on when the integral closure of I^ℓ is contained in the product $JI_{\ell-1}$, where $I_{\ell-1}$ is the intersection of the primary components of I of height $\leq \ell - 1$. In the case that R is also Gorenstein, we give results on when the integral closure of $I^{\ell-1}$ is contained in J.

1. Introduction

In this paper, all rings are assumed to be commutative and Noetherian with identity.

The theorem of Briançon and Skoda was first proved in an analytic setting. Namely, let $O_n = \mathbb{C}\{z_1, \ldots, z_n\}$ be the ring of convergent power series in n variables. Let $f \in O_n$ be a non-unit (i.e. f vanishes at the origin), and let $J(f) = (\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n})O_n$ be the Jacobian ideal of f. Then one can see that $f \in \overline{J(f)}$, the integral closure of $J(f)$, and in particular there is an integer k such that $f^k \in J(f)$. John Mather raised the following question: Does there exist an integer k that works for all non-units f?

Briançon and Skoda first answered this question affirmatively by proving that the n^{th} power of f lies in $J(f)$. This is an immediate result of the following theorem:

Theorem 1.1. ([5]) *Let $I \subseteq O_n$ be an ideal generated by ℓ elements. Then for all $w \geq 0$,*

$$\overline{I^{\ell+w}} \subseteq I^{w+1}.$$

Since $f \in \overline{J(f)}$, $f^n \in \overline{J(f)}^n \subseteq \overline{J(f)^n} \subseteq J(f)$, by applying the Briançon and Skoda theorem for $I = J(f)$, which has at most n generators (taking w to be zero). Hence $f^n \in J(f)$ and Mather's question is answered.

Lipman and Sathaye proved that this purely algebraic result can be extended to arbitrary regular local rings as follows:

Theorem 1.2. ([9]) *Let $(R, \mathfrak{m})$ be a regular local ring and suppose that I is an ideal of R generated by ℓ elements. Then for all $w \geq 0$,*

$$\overline{I^{\ell+w}} \subseteq I^{w+1}.$$

Lipman and Teissier were partially able to extend this theorem to pseudo-rational rings [10], while Aberbach and Huneke were able to prove the theorem for F-rational rings and rings of F-rational type in the equicharacteristic case [1].

Initially motivated by trying to understand the relationship between the Cohen-Macaulayness of the Rees ring of I, $R[It]$, and the associated graded ring of I, $\mathrm{gr}_I R$, various authors (see, e.g.,

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[1],[2],[3],[4],[6],[8],[12]) have studied the coefficients involved in the Briançon-Skoda Theorem. More specifically, if $J = (a_1, \ldots, a_\ell)$ is a minimal reduction of I and $z \in \overline{I^\ell} = \overline{J^\ell}$, then when we write $z = \sum_{i=1}^{\ell} r_i a_i$, we may ask where the coefficients r_i lie. For instance, can we say that they lie in I? Heuristically, when $\mathrm{ht}(I) < \ell$, there is reason to have this occur. One such result is Theorem 3.6 of [2]. We are able to substantially reduce the necessary hypotheses needed in that paper. Explicitly, we prove the following result (see the next section for the definition of $I_{\ell-1}$):

Theorem 3.9. *Let $(R, \mathfrak{m})$ be an F-rational Cohen-Macaulay local ring and $I \subseteq R$ be an ideal of analytic spread ℓ and of height $g < \ell$. If J is any reduction of I then $\overline{I^\ell} \subseteq JI_{\ell-1}$.*

We are also interested in reducing the power ℓ of I in $\overline{I^\ell} \subseteq J$. This is possible in Gorenstein rings, with the aid of local duality, using ideas first utilized in [7]. More precisely, we improve on Theorem 4.1 of [1], due to Aberbach and Huneke, to show:

Theorem 4.2. *Let $(R, \mathfrak{m})$ be an F-rational Gorenstein local ring of dimension d and characteristic $p > 0$. Suppose that I is an ideal of height g and analytic spread $\ell > g$. Assume that $I = I_{\ell-1}$ and that R/I has depth at least $d - \ell + 1$. Then for any reduction J of I, we have $\overline{I^{\ell-1}} \subseteq J$.*

2. Preliminary results

In this section, we review some of the definitions and results that will be used in this paper.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. An ideal $J \subseteq I$ is a *reduction* of I if there exists an integer n such that $JI^n = I^{n+1}$ [11]. The least such integer is the *reduction number* of I with respect to J. A reduction J of I is called a *minimal reduction* if J is minimal with respect to inclusion among reductions. When $(R, \mathfrak{m})$ is local with infinite residue field, every minimal reduction J of I has the same number of minimal generators. This number is called the *analytic spread* of I, denoted by $\ell(I)$, and we always have the inequalities $\mathrm{ht}(I) \leq \ell(I) \leq \dim R$. The *analytic deviation* of I, denoted by $ad(I)$, is the difference between the analytic spread of I and the height of I, i.e. $ad(I) = \ell(I) - \mathrm{ht}(I)$. We also define I^{un} to be the intersection of the minimal primary components of the ideal I (under this definition I^{un} has no embedded components but may have components of different heights or dimension).

An element x of R is said to be in the *integral closure* of I, denoted by $\overline{I}$, if x satisfies an equation of the form $x^k + a_1 x^{k-1} + \cdots + a_k = 0$ where $a_i \in I^i$ for $1 \leq i \leq k$. If an ideal $J \subseteq I$ is a reduction, then $\overline{J} = \overline{I}$.

Let R be a Noetherian ring of prime characteristic $p > 0$ and let q be a varying power of p. Denote by R^o the complement of the union of the minimal primes of R and let I be an ideal of R. Define $I^{[q]} = (i^q; i \in I)$, the ideal generated by the q^{th} powers of all the elements of I. The *tight closure* of I is the ideal $I^* = \{x \in R;\ \text{for some } c \in R^o, cx^q \in I^{[q]},\ \text{for } q >> 0\}$. We always have that $I \subseteq I^* \subseteq \overline{I}$. If $I^* = I$ then the ideal I is said to be *tightly closed*. We say that elements $x_1, \ldots, x_n$ of R are *parameters* if the height of the ideal generated by them is at least n (we allow them to be the whole ring, in which case the height is said to be ∞). The ring R is said to be *F-rational* if the ideals generated by parameters are tightly closed.

The remaining ingredients of this section are from [1] and [2].

Definition 2.1. ([2], Definition 2.10) Let R be a Noetherian local ring and let I be an ideal of height g. We say that a reduction $J = (a_1, \ldots, a_\ell)$ of I is generated by a *basic generating set* if for all prime ideals P containing I such that $i = \mathrm{ht}(P) \leq \ell$, $(a_1, \ldots, a_i)_P$ is a reduction of I_P.

When the residue field of R is infinite, there always exist such basic generating sets, and furthermore, $\mathrm{ht}((a_1, \dots, a_i)I^n : I^{n+1} + I) \geq i + 1$ for $n \gg 0$.

Proposition 2.2. ([1], Proposition 3.2) *Let $(R, \mathfrak{m})$ be an equidimensional and catenary local ring with infinite residue field and let $I \subseteq R$ be an ideal of height g and analytic spread ℓ. Let $J \subseteq I$ be a minimal reduction of I. We assume that $J = (a'_1, \dots, a'_\ell)$ is generated by a basic generating set as in Definition 2.1 above. Let N and w be fixed integers, and suppose that for $g + 1 \leq i \leq \ell$, we are given finite sets of primes $\Lambda_i = \{Q_{ji}\}$ all containing I and of height i. Then there exist elements $a_1, \dots, a_\ell$ and $t_{g+1}, \dots, t_\ell$ such that the following conditions hold. (We set $t_i = 0$ for $i \leq g$ for convenience).*

1. *$a_i \equiv a'_i$ modulo I^2.*
2. *For $g + 1 \leq i \leq \ell$, $t_i \in m^N$.*
3. *$b_1, \dots, b_g, b_{g+1}, \dots, b_\ell$ are parameters, where $b_i = a_i + t_i$.*
4. *The images of $t_{g+1}, \dots, t_\ell$ in R/I are part of a system of parameters.*
5. *There is an integer M such that $t_{i+1} \in (J_i^n I^M : I^{M+n})$ for all $0 \leq n \leq w + \ell$ where $J_i = (a_1, \dots, a_i)$.*
6. *$t_{i+1} \notin \cup_j Q_{ji}$, the union being over the primes in Λ_i.*

Remark 2.3. We have altered the statement made in part 4 of Proposition 2.2 from that of the original, but the given statement holds.

Proposition 2.2 allows us to choose a parameter ideal, $\mathfrak{m}$-adically as close to I as desired, for which a sort of Briançon-Skoda result applies. Specifically:

Theorem 2.4. ([1], Theorem 3.3) *Let $(R, \mathfrak{m})$ be an equidimensional and catenary local ring of characteristic p having an infinite residue field. Let I be an ideal of analytic spread ℓ and positive height g. Let J be a minimal reduction of I. Fix w and $N \geq 0$. Choose a_i and t_i as in Proposition 2.2. Set $\mathfrak{A} = B_\ell = (b_1, \dots, b_g, \dots, b_\ell)$. Then*

$$\overline{I^{\ell+w}} \subseteq (\mathfrak{A}^{w+1})^*.$$

One of our main goals in this paper is to generalize the next theorem, which is due to Aberbach and Huneke. We recall that an ideal I satisfies a property generically if I_p satisfies that same property for every minimal prime P of I.

Theorem 2.5. ([2], Theorem 3.6) *Let $(R, \mathfrak{m})$ be an excellent F-rational local ring, $I \subseteq R$ an ideal with reduction J. Let $g = ht(I) < \ell = \ell(I)$. Suppose that*

- *R/I is equidimensional.*
- *R/I^{un} satisfies $S_{\ell-g-1}$, and*
- *I is generically of reduction number at most one.*

Then $\overline{I^\ell} \subseteq JI^{un}$.

In particular if R/I is equidimensional, I generically has reduction number at most one and I has analytic deviation 2, then $\overline{I^\ell} \subseteq JI^{un}$.

3. The main theorem

We will show that Serre's condition $S_{\ell-g-1}$ on R/I^{un}, and the assumption that I is an ideal generically of reduction number at most one in Theorem 2.5 of Aberbach and Huneke are not necessary. We will also show that I^{un} may be replaced by a (potentially) smaller ideal, which in many instances, also allows us to remove the hypothesis that I is unmixed.

Lemma 3.1. *Let R be a Noetherian ring, J an ideal of R, and x an element of R. Then there exists a positive integer M such that $J :_R x^\infty = J :_R x^M$.*

Proof. $J :_R x \subseteq J :_R x^2 \subseteq \dots$ is an increasing sequence of ideals in the Noetherian ring R, so there exists M such that $J :_R x^{M+k} = J :_R x^M$ for all $k \geq 0$. Hence $J :_R x^\infty = \bigcup_{i \in \mathbb{N}} (J :_R x^i) = J :_R x^M$. $\square$

Remark 3.2. After renaming x^M back to x, we can assume that $J :_R x^\infty = J :_R x$.

Proposition 3.3. *Let R be a Noetherian ring and I an ideal of R. Let t_1 be an element of R such that its image is regular in R/I. Then for any elements $t_2, \dots, t_n$ of R, there exist elements $u_2, \dots, u_n$ of R such that for all $2 \leq i \leq n$, u_i is a power of t_i, and $(I, u_2^2, \dots, u_n^2) : t_1 \subseteq (I, u_2, \dots, u_n)$.*

Proof. Let s_1 be the image of t_1 in $S = R/I$ and set $S_1 = S/(s_1)$.

By Lemma 3.1 and Remark 3.2, we can pick $u_2 \in R$, a power of t_2, such that its image s_2 in S satisfies: $0 :_{S_1} s_2^\infty = 0 :_{S_1} s_2$. For $3 \leq i \leq n$, pick $u_i \in R$, a power of t_i, such that its image s_i in S satisfies

$$(s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^\infty = (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i.$$

We claim that for $2 \leq i \leq n$, if $w \in (s_2^2, \dots, s_i^2) :_S s_1$, there exists v_i such that

$$w - v_i s_i \in (s_2^2, \dots, s_{i-1}^2) :_S s_1.$$

To prove the claim, let w be in $(s_2^2, \dots, s_i^2) :_S s_1$, then one can write that

$$s_1 w = \alpha_2 s_2^2 + \dots + \alpha_i s_i^2. \tag{3.1}$$

Thus $\alpha_2 s_2^2 + \dots + \alpha_i s_i^2 = 0$ in S_1, which implies that

$$\alpha_i \in (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^2 \subseteq (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i^\infty = (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i.$$

Hence $\alpha_i \in (s_2^2, \dots, s_{i-1}^2) :_{S_1} s_i$, or equivalently $\alpha_i s_i \in (s_1, s_2^2, \dots, s_{i-1}^2)S$.

Write $\alpha_i s_i = v_i s_1 + x_i$ where $x_i \in (s_2^2, \dots, s_{i-1}^2)$. We get $\alpha_i s_i^2 = v_i s_1 s_i + x_i s_i$. Replacing this expression for $\alpha_i s_i^2$ back into (3.1), and combining the terms involving s_1, we see that $s_1(w - v_i s_i) \in (s_2^2, \dots, s_{i-1}^2)$, which implies that $w - v_i s_i \in (s_2^2, \dots, s_{i-1}^2) :_S s_1$, as desired.

Therefore, we can conclude that if $w \in (s_2^2, \dots, s_n^2) :_S s_1$, there exist $v_2, \dots, v_n$ such that $s_1(w - v_n s_n - \dots - v_2 s_2) = 0$ in S.

But s_1 is a nonzerodivisor in S, so $w - v_n s_n - \dots - v_2 s_2 = 0$ in S, implying that $w \in (s_2, \dots, s_n)S$. Therefore $(s_2^2, \dots, s_n^2) :_S s_1 \subseteq (s_2, \dots, s_n)S$, or equivalently, $(I, u_2^2, \dots, u_n^2) : t_1 \subseteq (I, u_2, \dots, u_n)R$ as desired. $\square$

Definition 3.4. If I has height g, then given $k \geq g$, set $S_k = R \setminus \bigcup P$ where the union is taken over all primes P that are associated to I and such that ht $(P) \leq k$. We define $I_k = I S_k^{-1} R \cap R$. This means that given a primary decomposition of I, I_k is the intersection of the primary components of I of height $\leq k$.

Let $(R, \mathfrak{m})$ be an equidimensional and catenary local ring with infinite residue field and let $I \subseteq R$ be an ideal of height g and analytic spread ℓ. Let $J \subseteq I$ be a minimal reduction of I. We assume that $J = (a'_1, \dots, a'_\ell)$ is generated by a basic generating set as in Definition 2.1. Let N and w be fixed integers, and suppose that for $g + 1 \leq i \leq \ell$ we are given finite sets of primes $\Lambda_i = \{Q_{ji}\}$ all containing I and of height i.

Proposition 3.5. *With the above assumptions, there exist elements $a_1, \ldots, a_\ell$ generating J and $t_{g+1}, \ldots, t_\ell$ of R such that conditions 1 through 6 of Proposition 2.2 hold.*

In addition, we can choose the elements $t_{g+1}, \ldots, t_\ell$ in R such that

$$(I_{\ell-1}, t_{g+1}^2, \ldots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{\ell-1}, t_{g+1}, \ldots, t_{\ell-1}).$$

Proof. Pick elements $a_1, \ldots, a_\ell$ and $t_{g+1}, \ldots, t_\ell$ in R as in Proposition 2.2. If necessary, replace $t_{g+1}, \ldots, t_{\ell-1}$ by higher powers so that

$$(I_{\ell-1}, t_{g+1}^2, \ldots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{\ell-1}, t_{g+1}, \ldots, t_{\ell-1}).$$

This is possible since on one hand properties 1-6 of Proposition 2.2 remain true after replacing $t_{g+1}, \ldots, t_{\ell-1}$ by higher powers. And on the other hand, we can apply Proposition 3.3 once we check that t_ℓ is a nonzerodivisor in $R/I_{\ell-1}$. But this is true by property 6 of Proposition 2.2 if we set $\Lambda_{\ell-1}$ to be any finite set of height $\ell - 1$ primes whose union contains all associated primes of I with height at most $\ell - 1$. ☐

A few more results are needed before we can give a proof to our main theorem in this section.

Let $(R, \mathfrak{m})$ be an equidimensional and catenary local ring of characteristic p, having an infinite residue field. Let I be an ideal of analytic spread ℓ and positive height g. Let J be a minimal reduction of I. Fix integers w and $N \geq 0$, and choose $a_1, \ldots, a_\ell$ and $t_{g+1}, \ldots, t_\ell$ as in Proposition 2.2. For $i = 1, \ldots, \ell$, set $b_i = a_i + t_i$ ($t_i = 0$ for $i \leq g$), $J_i = (a_1, \ldots, a_i)$, and $B_i = (b_1, \ldots, b_i)$.

Lemma 3.6. *With the above assumptions, there exists an element $c \in I^M \cap R^0$ (where M is the integer from condition 5 of Proposition 2.2) such that the following conditions hold:*

1. *For any $g + 1 \leq j \leq \ell$ and $1 \leq k \leq w + \ell$, $ct_j^q I^{kq} \subseteq J_{j-1}^{kq}$, for any power q of p.*
2. *For all $g \leq i \leq \ell$ and $0 \leq r \leq w$, we have $c^{i-g} J_i^{(i+r)q} \subseteq (B_i^{r+1})^{[q]}$, for any power q of p.*

Proof. This lemma combines useful facts that were presented in the proof of Theorem 2.4. For their proofs, refer to the proof of Theorem 3.3 in [1]. ☐

The next result is a generalization of Lemma 4.3 in [1].

Lemma 3.7. *Under the above assumptions, assume that $g < \ell$. Let m be 0 or -1. Then for all $g + 1 \leq j \leq \ell$, we have*

$$t_j \, \overline{I^{\ell+m}} \subseteq (B_{j-1}^{\ell-j+m+2})^*.$$

Proof. Fix an integer $w \geq \ell - g \geq 0$ and let $z \in \overline{I^{\ell+m}}$. Then there exists an element $d \in R^0$ such that $dz^q \in I^{(\ell+m)q}$, for all powers q of p. Also, choose $c \in I^M \cap R^0$ satisfying the conclusions of Lemma 3.6.

For $g + 1 \leq j \leq \ell$, $dc^{j-g} t_j^q z^q \in t_j^q c^{j-g} \, I^{(\ell+m)q} = c^{j-g-1} c t_j^q \, I^{(\ell+m)q}$. Apply Lemma 3.6(1) and (2) to obtain $dc^{j-g} t_j^q z^q \in c^{j-1-g} \, J_{j-1}^{(\ell+m)q} \subseteq (B_{j-1}^{\ell-j+m+2})^{[q]}$ when taking $g \leq i = j - 1 \leq \ell$ and $0 \leq r = \ell - j + m + 1 \leq \ell - g \leq w$.

Hence, $dc^{j-g} t_j^q z^q \in (B_{j-1}^{\ell-j+m+2})^{[q]}$ which implies that $t_j z \in (B_{j-1}^{\ell-j+m+2})^*$, since the element dc^{j-g} is in R^0. Therefore, we conclude that $t_j \, \overline{I^{\ell+m}} \subseteq (B_{j-1}^{\ell-j+m+2})^*$, for $m = 0$ or -1. ☐

Remark 3.8. In Theorem 2.4 and Lemmas 3.6 and 3.7, if one replaces any $b_i = a_i + t_i$ by $a_i + t_i^2$, the conclusions remain unchanged. This is true because by raising any t_i to a higher power, conditions 1 through 6 of Proposition 2.2 still hold.

The next theorem generalizes Aberbach and Huneke's Theorem 3.6 of [1], stated here as Theorem 2.5. It shows that Serre's condition, the assumption that R/I is equidimensional and the generic reduction number hypothesis are superfluous. In the proof that we present, we make the appropriate modifications to Aberbach and Huneke's proof of Theorem 2.5.

Theorem 3.9. *Let $(R, \mathfrak{m})$ be an F-rational Cohen-Macaulay local ring (e.g., an excellent F-rational local ring), $I \subseteq R$ an ideal of analytic spread ℓ and of height $g < \ell$ and let J be any reduction of I. Then $\overline{I^\ell} \subseteq JI_{\ell-1}$.*

Proof. Without loss of generality, assume that R has an infinite residue field and $J = (a_1, \dots, a_\ell)$ is a minimal reduction generated by a basic generating set.

Fix an integer N and choose $t_{g+1}, \dots, t_\ell \in m^N$ as in Proposition 3.5 (here we set $w = 0$). Hence, $t_{g+1}, \dots, t_\ell$ satisfy the conditions of Proposition 2.2 as well as the inclusion

$$(I_{\ell-1}, t_{g+1}^2, \dots, t_{\ell-1}^2) :_R t_\ell \subseteq (I_{\ell-1}, t_{g+1}, \dots, t_{\ell-1}).$$

Set $b_k = a_k + t_k^2$ (we set $t_k = 0$ for $k \leq g$). By Theorem 2.4, $\overline{I^\ell} \subseteq (a_1, \dots, a_g, b_{g+1}, \dots, b_\ell)$. Given $z \in \overline{I^\ell}$, we may write $z = r_1 a_1 + \cdots + r_g a_g + r_{g+1} b_{g+1} + \cdots + r_\ell b_\ell$, where $r_i \in R$, for $1 \leq i \leq \ell$. We aim to show that $r_i \in I_{\ell-1} + m^N$, for all $i = 1, \dots, \ell$.

For $1 \leq i \leq \ell$,

$$\begin{aligned} t_\ell r_i b_i &\in t_\ell \, (\overline{I^\ell}, b_1, \dots, \widehat{b_i}, \dots, b_\ell) \\ &\subseteq ((a_1, \dots, a_g, b_{g+1}, \dots, b_{\ell-1})^2)^* + (b_1, \dots, \widehat{b_i}, \dots, b_\ell), \text{ by Lemma 3.7} \\ &\subseteq (b_1, \dots, b_i^2, \dots, b_\ell). \end{aligned}$$

By combining the terms involving b_i, we conclude that

$$\begin{aligned} t_\ell r_i &\in (b_i) + (b_1, \dots, \widehat{b_i}, \dots, b_\ell) : b_i \\ &= (b_i) + (b_1, \dots, \widehat{b_i}, \dots, b_\ell), \text{ since } b_1, \dots, b_\ell \text{ is a regular sequence} \\ &\subseteq (J, t_{g+1}^2, \dots, t_\ell^2) \\ &\subseteq (I_{\ell-1}, t_{g+1}^2, \dots, t_\ell^2). \end{aligned}$$

Now combine the terms containing t_ℓ to obtain that

$$\begin{aligned} r_i &\in (t_\ell) + (I_{\ell-1}, t_{g+1}^2, \dots, t_{\ell-1}^2) : t_\ell \\ &\subseteq (t_\ell) + (I_{\ell-1}, t_{g+1}, \dots, t_{\ell-1}), \text{ by Proposition 3.5.} \end{aligned}$$

Hence, $r_i \in (I_{\ell-1}, t_{g+1}, \dots, t_\ell) \subseteq I_{\ell-1} + m^N$, for all $i = 1, \dots, \ell$.

We conclude that $z \in I_{\ell-1}(a_1, \dots, a_g) + I_{\ell-1}(b_{g+1}, \dots, b_\ell) + m^N \subseteq I_{\ell-1}(a_1, \dots, a_\ell) + m^N$.

As N was arbitrary, the Krull intersection theorem gives that $z \in JI_{\ell-1}$, finishing the proof of the theorem. □

Remark 3.10. When R/I is equidimensional, $I_{\ell-1} \subseteq I^{un}$, so Theorem 3.9 implies in this case that $\overline{I^\ell} \subseteq JI^{un}$. Hence Theorem 3.9 is a generalization of Aberbach and Huneke's Theorem 2.5, but removes the hypotheses involving Serre's condition on R/I^{un}, the generic reduction number of I, and the assumption that R/I is equidimensional.

4. A theorem for F-rational Gorenstein rings

In this section, we are interested in the cases where the power ℓ of I in the inclusion $\overline{I^\ell} \subseteq J$ (where J is a reduction of I), can be lowered. A cancellation theorem due to Huneke [7] inspired the main idea behind the proof of our next result. In particular, we extend another theorem of Aberbach and Huneke that states:

Theorem 4.1. ([1], Theorem 4.1) *Let $(R, \mathfrak{m})$ be an F-rational Gorenstein local ring of dimension d and having positive characteristic. Suppose that I is an ideal of height g and analytic spread $\ell > g$, with R/I Cohen-Macaulay. Then for any reduction J of I, $\overline{I^{\ell-1}} \subseteq J$.*

We extend Theorem 4.1 of Aberbach and Huneke in the following way:

Theorem 4.2. *Let (R, m) be an F-rational Gorenstein local ring of dimension d and characteristic $p > 0$. Suppose that I is an ideal of height g and analytic spread $\ell > g$. Assume that $I = I_{\ell-1}$ and that R/I has depth at least $d - \ell + 1$. Then for any reduction J of I, we have $\overline{I^{\ell-1}} \subseteq J$.*

In particular, if $\ell = d$ and $I = I_{\ell-1}$, then $\overline{I^{\ell-1}} \subseteq J$.

Proof. The proof is a modification of the proof of Theorem 4.1 presented in [1].

Without loss of generality, we may assume that R has an infinite residue field and that J is a minimal reduction of I. Fix an integer $N \geq 0$, and set $\Lambda_{\ell-1}$ to be any finite set of primes of R of height $\ell - 1$ such the the union contains all associated primes of I of height at most $\ell - 1$. Choose $a_1, \ldots, a_\ell$ and $t_{g+1}, \ldots, t_\ell$ as in Proposition 2.2 (here we set $w = 0$).

For $1 \leq i \leq \ell$, let $b_i = a_i + t_i^2$ (with $t_i = 0$ for $i \leq g$), $J_i = (a_1, \ldots, a_i)$ and $B_i = (b_1, \ldots, b_i)$.

By our choice of $\Lambda_{\ell-1}$, t_ℓ is a nonzerodivisor in $R/I_{\ell-1} = R/I$. Since depth $R/I \geq d - \ell + 1$, we can pick elements $x_{\ell+1}, \ldots, x_d$ in R such that $b_1, \ldots, b_\ell, x_{\ell+1}, \ldots, x_d$ is a regular sequence in R, and such that $t_\ell, x_{\ell+1}, \ldots, x_d$ is a regular sequence in R/I.

By Proposition 3.3 we can replace $t_{g+1}, \ldots, t_{\ell-1}$ by higher powers of themselves so that

$$(I, t_{g+1}^2, \ldots, t_{\ell-1}^2, x_{\ell+1}, \ldots, x_d) : t_\ell^2 \subseteq (I, t_{g+1}, \ldots, t_{\ell-1}, x_{\ell+1}, \ldots, x_d), \tag{4.1}$$

where to obtain this inclusion we use that t_ℓ^2 is regular modulo $(I, x_{\ell+1}, \ldots, x_d)$.

Set $\mathfrak{A} = B_\ell + (x_{\ell+1}, \ldots, x_d)$, $D = B_{\ell-1} : t_\ell^2$ and $K = B_{\ell-1} + (x_{\ell+1}, \ldots, x_d)$. Note that $K : b_\ell = K$ since the elements involved form a regular sequence in R.

Let $Q = (I, t_{g+1}, \ldots, t_{\ell-1}, x_{\ell+1}, \ldots, x_d) + K : D$. We claim that $\mathfrak{A} : t_\ell^2 \subseteq Q$. Let $t_\ell^2 u \in \mathfrak{A}$ and write

$$t_\ell^2 u = w + vb_\ell = w + va_\ell + vt_\ell^2, \tag{4.2}$$

where $w \in K$. Then $t_\ell^2(u - v) \in B_{\ell-1} + (a_\ell, x_{\ell+1}, \ldots, x_d)$, and hence

$$\begin{aligned} u - v \in (B_{\ell-1} + (a_\ell, x_{\ell+1}, \ldots, x_d)) : t_\ell^2 &\subseteq (I, t_{g+1}^2, \ldots, t_{\ell-1}^2, x_{\ell+1}, \ldots, x_d) : t_\ell^2 \\ &\subseteq (I, t_{g+1}, \ldots, t_{\ell-1}, x_{\ell+1}, \ldots, x_d), \text{by (4.1)}. \end{aligned}$$

Thus, $u - v \in Q$. To show that $u \in Q$, it suffices to show that $v \in K : D \subseteq Q$. Let $d \in D$ and consider dv. By (4.2), $dt_\ell^2 u = dw + dvb_\ell$. But as $dt_\ell^2 \in B_{\ell-1}$, $dvb_\ell \in K$. Therefore, $dv \in K : b_\ell = K$, as noted above. Consequently, $dv \in K$ and $v \in K : D \subseteq Q$. Hence, $u \in Q$ and this proves the claim that $\mathfrak{A} : t_\ell^2 \subseteq Q$. In particular it proves that $\mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_\ell^2)$.

Next, we show that $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q$. First, recall that $t_\ell^2 \overline{I^{\ell-1}} \subseteq B_{\ell-1}$, by Lemma 3.7. Thus, $\overline{I^{\ell-1}} \subseteq D$, and hence $\overline{I^{\ell-1}}(K : D) \subseteq D(K : D) \subseteq K \subseteq \mathfrak{A}$. Moreover, $I\,\overline{I^{\ell-1}} \subseteq \overline{I^\ell} \subseteq \mathfrak{A}^* = \mathfrak{A}$, by Theorem 2.4 and the fact that R is F-rational.

For $g+1 \leq j \leq \ell-1$, Lemma 3.7 implies that

$$t_j\, \overline{I^{\ell-1}} \subseteq ((B_{j-1}^2)^{\ell-j+1})^* \subseteq B_{j-1}^* = B_{j-1} \subseteq \mathfrak{A}.$$

Consequently, $(t_{g+1}, \dots, t_{\ell-1})\, \overline{I^{\ell-1}} \subseteq \mathfrak{A}$. Therefore, we have proved that $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q$.

Finally, $\overline{I^{\ell-1}} \subseteq \mathfrak{A} : Q \subseteq \mathfrak{A} : (\mathfrak{A} : t_\ell^2) = (\mathfrak{A}, t_\ell^2)$, by local duality. Hence,

$$\overline{I^{\ell-1}} \subseteq (J, t_{g+1}^2, \dots, t_{\ell-1}^2, t_\ell^2, x_{\ell+1}, \dots, x_d) \subseteq J + m^N.$$

An application of the Krull intersection theorem proves that $\overline{I^{\ell-1}} \subseteq J$. □

Remark 4.3. If I is unmixed and equidimensional (e.g., R/I is CM) in an F-rational Gorenstein ring then $I = I_{\ell-1}$. Therefore, Theorem 4.2 is a generalization of Theorem 4.1.

Question 4.4. Suppose that $\ell - g \geq 2$. Is it possible to improve on Theorem 4.2 to obtain that $\overline{I^{\ell-2}} \subseteq J$? If, in the proof of Theorem 4.2, we could show that, in fact, $\overline{I^{\ell-1}} \subseteq B_\ell + (x_{\ell+1}, \dots, x_d)$, then one could extend the result.

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Mathematics Department, University of Missouri, Columbia, MO 65211 USA